

## Beta Advection-Diffusion Model

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### Background.

The current migration sub-model used in Ken Newman's State Space Model (SSM) is loosely called a Beta Advection-Diffusion model. The SSM estimates a single parameter (*Move\_Alpha*) that completely describes the movement of fish over all time periods and locations. At the August 27, 1998 model committee meeting many members asked for an intuitive description of how this single parameter characterized fish migration. The purpose of this note is to define the Beta Advection-Diffusion model and describe its behavior. Another description is given in Ken's latest draft of his paper.

### The Model.

Let  $l_t$  be the current location of a fish (at time  $t$ ). Then the next location of that fish (at time  $t + 1$ ) is described by a Beta( $\alpha, \beta$ ) probability distribution where  $\beta$  is assumed constant (= 3.0) and  $\alpha$  is a function of *Move\_Alpha*, current location ( $l_t$ ), and current time ( $t$ ) as follows:

$$\alpha = (\textit{Move\_Alpha}) \cdot (\textit{Dist\_Scalar}) \cdot (\textit{Time\_Scalar})$$

where the distance scalar (*Dist\_Scalar*) represents how close the fish is to the natal stream and is given by

$$\textit{Dist\_Scalar} = \frac{l_t}{D}$$

( $D$  = maximum possible distance from the natal stream) and the time scalar (*Time\_Scalar*) represents how close the current time is to the last time period ( $T$ ) and is given by

$$\textit{Time\_Scalar} = \frac{T + 1 - t}{T + 1}.$$

The "1" in the numerator prevents a zero value for *Time\_Scalar* and the "1" in the denominator roughly offsets the "1" in the numerator. The net effect is that as a fish moves closer to the natal stream ( $l_t$  gets smaller) and as time increases, both scalars get smaller and the  $\alpha$  parameter of the Beta distribution gets smaller.

Model Properties.

The expected value of a Beta distributed random variable is  $\alpha/(\alpha + \beta)$ . Thus, as time increases and the  $\alpha$  parameter gets smaller, the expected new location moves toward the natal stream.

The expected step size for a fish located at  $l_t$  and time  $t$  is

$$E(\text{Step}) = l_t - \frac{\alpha}{\alpha + \beta}.$$

Since  $\beta$  is fixed and  $\alpha$  is a function of *Move\_Alpha*,  $l_t$ , and  $t$ , expected step size can be computed for all possible combinations of location and time. Or more generally, one can compute expected step size as a function of *Dist\_Scalar* and *Time\_Scalar*.

To get a better "feel" for this migration model, I created an Excel Spreadsheet to compute a table of expected step sizes for combinations of *Dist\_Scalar* (0 to 1 in 0.1 increments) and *Time\_Scalar* (also 0 to 1 in 0.1 increments) given *Move\_Alpha* and  $\beta$  (Table 1.) I plotted the resulting expected step size surface as a 3D plot (Fig. 1) and also plotted expected step size as a function of current location for five relative time values (0.1, 0.3, 0.5, 0.7, and 0.9; Fig. 2). The table values and plots update automatically whenever *Move\_Alpha* and  $\beta$  are changed.

The general migration pattern is the following: at any given time, there is an "Attracting Location" at which the expected step size is zero. Fish located to the right of this location (i.e., Current Location > Attracting Location) have a negative expected step size (i.e., they move toward the Attracting Location); fish located to the left of this location (i.e., Current Location < Attracting Location) have a positive expected step size (i.e., they also move toward the Attracting Location). As the season progresses (i.e., time gets bigger), the Attracting Location moves in the direction of the natal stream (i.e., the origin).

With  $\beta$  fixed at 3.0, increasing values of *Move\_Alpha* move the initial Attracting Location further from the origin. I created a table (Table 2) and graph (Fig. 3) to illustrate the relationship between Attracting Location and time for values of *Move\_Alpha* ranging from 3.0 to 15.0. When *Move\_Alpha* is about 5.0, the Attracting Location is near the origin for all times. When *Move\_Alpha* = 9.0, the Attracting Location starts out at about 0.67 and when current time reaches about 0.67 the Attracting Location is the origin. When *Move\_Alpha* = 15.0, the Attracting Location starts out at about 0.80 and when current time reaches 0.80 the Attracting Location is the origin.

Discussion (summary of email between Norris and Newman).

Norris: Although the Beta Advection-Diffusion model has some appealing properties (e.g., Attracting Location moves toward the origin as time increases), I find the property of an

Attracting Location early in the season somewhat disturbing because it implies a directed movement toward a specific location other than the natal stream. And for coastal stocks, such as the Humptulips, it implies that fish located north and south of the natal stream during the first part of the modeling period have directed movement toward different locations (i.e., there are two Attracting Locations, one north and one south of the natal stream).

Newman: Ken offered another interpretation. Early in the season, fish near the natal area are "free" to keep moving away from the natal area ...they're closing enuf that they've got time to "dawdle," while fish further away need to get moving home.

Norris: I'm having difficulty linking this type of movement model with my intuition about how salmon behave in the ocean. I think the initial fish distribution is determined by a combination of genetic factors and physical and biological oceanographic conditions. The genetic factors seem to put limits on the range of latitudes the fish are willing to inhabit, while oceanographic conditions determine the degree of patchiness, or aggregations, within those limits (e.g., due to prey, predators, temperature, salinity, olfactory cues, etc). Thus, I think it is possible for the initial distribution of a stock to have one or more concentrations. In our modeling effort I think we need a flexible distribution to account for skewed initial distributions.

Newman: Ken noted that another factor affecting initial fish distribution is size and time of release for hatchery fish. This also affects maturation rates for chinook. With the Humptulips stocks apparently "turning" south and north--a bimodal initial distribution could make some sense.

Norris: Given the initial distribution of the fish, I think the migration model should be founded, as much as possible, on what is known about individual fish behavior. Unfortunately, not much is known! And since we all have our own ideas about what that behavior is, I think we'll have to try several types of models to see which ones fit real data the best, and also make the most sense biologically. As you mentioned in one of your emails, the fact that the Beta Advection-Diffusion model predicts some unrealistically high migration rates late in the season may not be a problem because those unrealistic rates are only predicted for fish residing outside the realistic domain of the fish at that time. The same is true for the "Increasing" migration model I used to generate synthetic data--by the time the last time steps are reached, the daily rate is pretty high, but all the fish are already in the river.

Newman: Your key statement is "not much is known" -- which has kept me pointed towards doing largely data driven selection of models. If someone comes up with an alternative "module" for initial distribution, survival/harvest, and/or migration and it fits better for a relatively large set of stocks and cohorts than does a current configuration, then that's as good an argument as any to pick the alternative.

Table 1. Expected Step Size for values of relative location and time ( $Move\_Alpha = 8.735$ ;  $\beta = 3.0$ ).

Rel Loc	Relative Elapsed Time										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.1	0.13	0.11	0.09	0.07	0.05	0.03	0.00	-0.02	-0.04	-0.07	-0.10
0.2	0.17	0.14	0.12	0.09	0.06	0.03	-0.01	-0.05	-0.10	-0.14	-0.20
0.3	0.17	0.14	0.11	0.08	0.04	0.00	-0.04	-0.09	-0.15	-0.22	-0.30
0.4	0.14	0.11	0.08	0.05	0.01	-0.03	-0.08	-0.14	-0.21	-0.30	-0.40
0.5	0.09	0.07	0.04	0.00	-0.03	-0.08	-0.13	-0.20	-0.27	-0.37	-0.50
0.6	0.04	0.01	-0.02	-0.05	-0.09	-0.13	-0.19	-0.26	-0.34	-0.45	-0.60
0.7	-0.03	-0.05	-0.08	-0.11	-0.15	-0.20	-0.25	-0.32	-0.41	-0.53	-0.70
0.8	-0.10	-0.12	-0.15	-0.18	-0.22	-0.26	-0.32	-0.39	-0.48	-0.61	-0.80
0.9	-0.18	-0.20	-0.22	-0.25	-0.29	-0.33	-0.39	-0.46	-0.56	-0.69	-0.90
1	-0.26	-0.28	-0.30	-0.33	-0.36	-0.41	-0.46	-0.53	-0.63	-0.77	-1.00

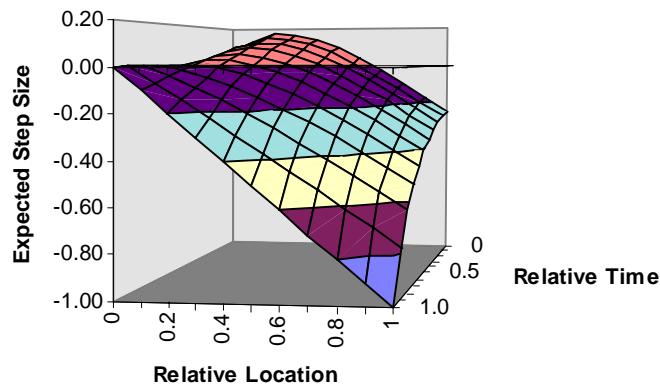


Fig. 1. Expected step size as a function of relative location (i.e.,  $Dist\_Scalar$ ) and relative time ( $Time\_Scalar$ ).

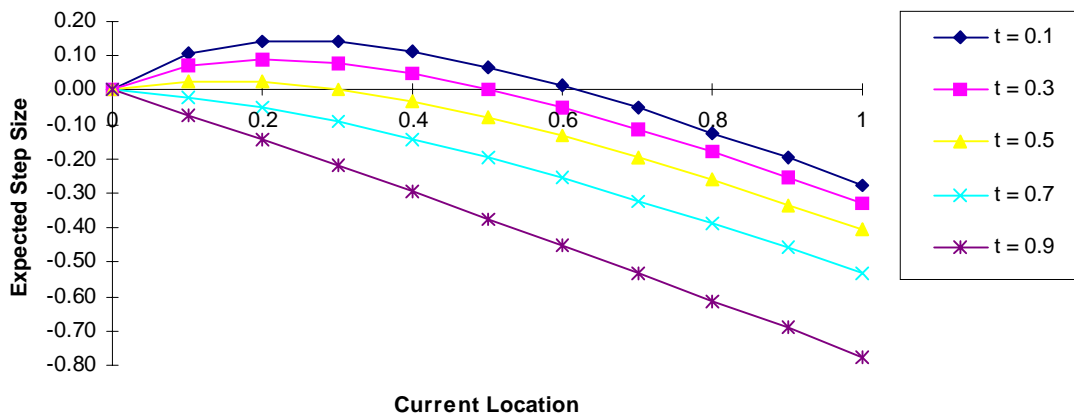


Fig. 2. Expected step size as a function of current location for five values of relative  $t$ .

**Table 2. "Attraction Location" by Relative Time and *Move\_Alpha*.**

Rel Time	<i>Move_Alpha</i>						
	3.00	5.00	7.00	9.00	11.00	13.00	15.00
<b>0</b>	0.00	0.40	0.57	0.67	0.73	0.77	0.80
<b>0.1</b>	-0.11	0.33	0.52	0.63	0.70	0.74	0.78
<b>0.2</b>	-0.25	0.25	0.46	0.58	0.66	0.71	0.75
<b>0.3</b>	-0.43	0.14	0.39	0.52	0.61	0.67	0.71
<b>0.4</b>	-0.67	0.00	0.29	0.44	0.55	0.62	0.67
<b>0.5</b>	-1.00	-0.20	0.14	0.33	0.45	0.54	0.60
<b>0.6</b>	-1.50	-0.50	-0.07	0.17	0.32	0.42	0.50
<b>0.7</b>	-2.33	-1.00	-0.43	-0.11	0.09	0.23	0.33
<b>0.8</b>	-4.00	-2.00	-1.14	-0.67	-0.36	-0.15	0.00
<b>0.9</b>	-9.00	-5.00	-3.29	-2.33	-1.73	-1.31	-1.00

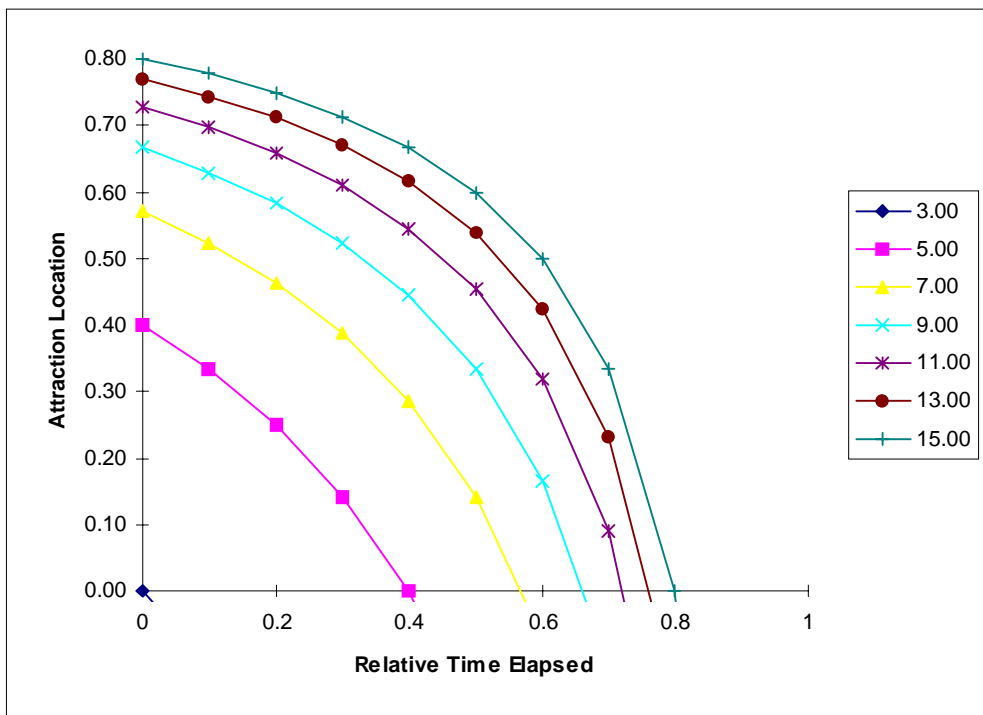


Fig. 3. "Attraction Location" as a function of relative time for several values of *Move\_Alpha*.